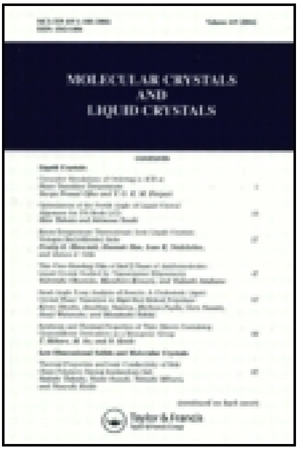

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Waveguide Propagation of Light in Cholesterics with Large Pitch

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The Green's function and waveguide propagation of electromagnetic field in cholesteric liquid crystals with a pitch being large compared to the wavelength is considered. This function is constructed using the solution of the Maxwell equations. The behavior in the far zone is analysed in detail. The periodic system is distinguished from an anisotropic medium by a discontinuity of the wave vector surface and a break of the beam vector surface. Trajectories of beams in such a medium are not plane. The forbidden zone corresponds to capture of beams and formation of a wave channel. Within this wave channel the Green's function asymptotic differs from $1/r$ behavior.

Keywords: Green function; cholesteric; large pitch; waveguide

1 INTRODUCTION

Cholesteric liquid crystals (CLC) are systems with unusual optical properties [1]. The reason is that a cholesteric is a locally uniaxial medium, with direction of the optical axis periodically varying in space. The problem of electromagnetic waves propagation in one-dimensional periodic media and among them in cholesterics is important both for the theory of optical properties of complicated systems and for application of CLC as systems of mapping an information.

Considerable study has been given to cholesterics with a pitch d compared with a wavelength λ. In this case diffractional phenomena, selective reflection of circularly polarized light, forbidden zones and the anomalously large optical activity are investigated in detail [2, 3]. At the same time cholesterics with a pitch exceeding a wavelength significantly ($d \gg \lambda$) and weakly twisted nematics attract particular interest. Although these systems appear to be more simple, nevertheless unusual optical effects are observed here too.

In order to study the optical properties of these systems it is convenient to investigate the Green's function of electromagnetic field. The point is that the Green's function is actually a field of a point source. Therefore it gives an information on a field, propagated in all directions inside a medium. By now for construction of the Green's function in the CLC a formal algorithm exists [4]. It presents a numerical procedure only and does not admit the receiving an explicit analytical solution.

The present work is devoted to study of the Green's function in CLC for a pitch exceeding a light wavelength significantly ($d \gg \lambda$). The solution contains two contributions due to existence of ordinary and extraordinary waves. For an ordinary wave an anisotropy and periodicity of cholesterics are not exhibited, and this contribution to the Green's function is the same, as in a usual isotropic medium. For an extraordinary wave at large distances a forbidden zone exists. This zone means a restriction on possible directions of wave vectors. The similar effect exists when studying the propagation of waves in ocean, troposphere (troposphere refraction), plasma and other stratified media [5]. The effect results from a turn of beams caused by the variation of a refraction index. For CLC a refraction index of an extraordinary wave is a periodic function. Therefore beams not only turn their trajectories, but they also are captured, and a planar wave guide channel is formed.

In the present work the forbidden zone and the capture of beams in the wave channel is investigated. It is shown, that in such a medium a beam does not lie in a plane and the asymptotics of the Green's function inside and outside of the wave channel are different.

2 GENERAL EQUATIONS

In cholesterics the optical axis is parallel to the unit vector of the director $\mathbf{n}$, which rotates along the axis of cholesteric $\mathbf{e}_z$:

$$\mathbf{n}(\mathbf{r}) = (\cos(\alpha z + \psi_0)\,;\ \sin(\alpha z + \psi_0)\,;\ 0)\,, \tag{1}$$

where $\alpha = \pi/d$, ψ_0 defines the direction of $\mathbf{n}$ at $z = 0$ plane. The optical properties of CLC are determined by the locally uniaxial tensor of permittivity [1, 2, 3]

$$\varepsilon_{\alpha\beta}(\mathbf{r}) = \varepsilon_{\perp}\delta_{\alpha\beta} + \varepsilon_a n_\alpha(\mathbf{r}) n_\beta(\mathbf{r})\,. \tag{2}$$

Here $\varepsilon_a = \varepsilon_{\parallel} - \varepsilon_{\perp}$; $\varepsilon_{\parallel}$, $\varepsilon_{\perp}$ are permittivities along and across $\mathbf{n}$.

The Maxwell equations in such a medium have the form

$$\operatorname{curl}\mathbf{E}(\mathbf{r}) = ik_0\mu\mathbf{H}(\mathbf{r})\,, \quad \operatorname{curl}\mathbf{H}(\mathbf{r}) = -ik_0\hat{\varepsilon}(z)\mathbf{E}(\mathbf{r})\,, \tag{3}$$

where $\mathbf{E}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$ are electric and magnetic vectors, $k_0 = \omega/c$, ω is a circular frequency, c is the velocity of light in vacuum. Hereinafter we suppose a magnetic permeability $\mu = 1$. The wave equation has the form

$$\left(\operatorname{curl}\operatorname{curl} - k_0^2\hat{\varepsilon}(z)\right)\mathbf{E}(\mathbf{r}) = 0\,. \tag{4}$$

The Green's function (field of a point source) obeys the equation

$$\left(\operatorname{curl}\operatorname{curl} - k_0^2\,\hat{\varepsilon}(z)\right)\widehat{T}(\boldsymbol{\rho} - \boldsymbol{\rho}_1, z, z_1) = \delta(\mathbf{r} - \mathbf{r}_1)\widehat{I}\,, \tag{5}$$

where $\boldsymbol{\rho} = (x, y)$, $\delta(\mathbf{r} - \mathbf{r}_1)$ is delta-function, $\widehat{I}$ is the unit matrix. The Green's function permits to solve the wave equation with an arbitrary right hand side $\mathbf{F}(\mathbf{r})$:

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0(\mathbf{r}) + \int d\mathbf{r}_1 \widehat{T}(\boldsymbol{\rho} - \boldsymbol{\rho}_1, z, z_1)\mathbf{F}(\mathbf{r}_1)\,, \tag{6}$$

where $\mathbf{E}_0(\mathbf{r})$ is a solution of the wave equation with a zero right hand side.

As $\hat{\varepsilon}(z)$ depends on the z variable only, we complete in Eq. (5) a two-dimensional Fourier transformation

$$\widehat{T}(\mathbf{q}; z, z_1) = \int d\boldsymbol{\rho}\exp(-i\mathbf{q}\boldsymbol{\rho})\widehat{T}(\boldsymbol{\rho}, z, z_1)\,. \tag{7}$$

Let's select directions of x and y axes: $\mathbf{x} \parallel \mathbf{q}$, $\mathbf{y} \perp \mathbf{q}$. We get

$$\widehat{L}(z)\widehat{T}(\mathbf{q}; z, z_1) = \delta(z - z_1)\widehat{I}, \tag{8}$$

where

$$\widehat{L}(z) = \begin{pmatrix} -\dfrac{\partial^2}{\partial z^2} - k_0^2\varepsilon_{xx} & -k_0^2\varepsilon_{xy} & iq\dfrac{\partial}{\partial z} \\ -k_0^2\varepsilon_{xy} & -\dfrac{\partial^2}{\partial z^2} + q^2 - k_0^2\varepsilon_{yy} & 0 \\ iq\dfrac{\partial}{\partial z} & 0 & q^2 - k_0^2\varepsilon_\perp \end{pmatrix}. \tag{9}$$

Thus, the Green's function calculation is reduced to the solution of the system of ordinary linear differential equations, with periodically varying coefficients.

3 GREEN'S FUNCTION

Note, that the third line and the third column of the $\widehat{L}(z)$ matrix do not contain the second order derivative over z. Therefore in the system (8) it is convenient to exclude elements of the third line and the third column of the $\widehat{T}(\mathbf{q}; z, z_1)$ matrix. It gives the possibility to reduce the system of nine equations (8) to a system of four equations. In this case the order of equations do not increase. We have

$$-\frac{\partial^2}{\partial z^2}\widehat{G}(z, z_1) + \widehat{B}(z)\widehat{G}(z, z_1) = \delta(z - z_1)\widehat{I}, \tag{10}$$

where

$$\widehat{B}(z) = k_0^2 \begin{pmatrix} -\varepsilon_{xx}(1 - \mathcal{H}) & -\varepsilon_{xy}(1 - \mathcal{H}) \\ -\varepsilon_{xy} & -\varepsilon_{yy} + \varepsilon_\perp\mathcal{H} \end{pmatrix}, \tag{11}$$

$\mathcal{H} = q^2/k_0^2\varepsilon_\perp$. If the function $\widehat{G}(z, z_1)$ is known it is easy to obtain the Green's function $\widehat{T}(\mathbf{q}; z, z_1)$, using the system (8)

$$\begin{aligned} &T_{j1}(z, z_1) = (1 - \mathcal{H})G_{j1}(z, z_1), \; T_{j2}(z, z_1) = G_{j2}(z, z_1), \\ &T_{j3}(z, z_1) = \frac{iq}{q^2 - k_0^2\varepsilon_\perp}\frac{\partial T_{j1}(z, z_1)}{\partial z_1}, \\ &T_{3j}(z, z_1) = -\frac{iq}{q^2 - k_0^2\varepsilon_\perp}\frac{\partial T_{1j}(z, z_1)}{\partial z}, \qquad j = 1, 2; \\ &T_{33}(z, z_1) = \frac{\delta(z - z_1)}{q^2 - k_0^2\varepsilon_\perp} + \frac{q^2}{(q^2 - k_0^2\varepsilon_\perp)^2}\frac{\partial^2 T_{11}(z, z_1)}{\partial z \partial z_1}. \end{aligned} \tag{12}$$

The boundary condition for the Green's function $\widehat{T}$ in coordinate representation is a condition of radiation. It leads to the corresponding boundary condition for the function $\widehat{G}(z, z_1)$ in the system (10). Then we can write for the function $\widehat{G}(z, z_1)$ [6]

$$\widehat{G}(z, z_1) = \begin{cases} \widehat{V}_1(z)\widehat{V}_1^{-1}(z_1)\widehat{W}^{-1}(z_1), & z \geqslant z_1 \\ \widehat{V}_2(z)\widehat{V}_2^{-1}(z_1)\widehat{W}^{-1}(z_1), & z < z_1 \end{cases} \tag{13}$$

where $\widehat{W}(z) = \widehat{V}_2'(z)\widehat{V}_2^{-1}(z) - \widehat{V}_1'(z)\widehat{V}_1^{-1}(z)$, $\lim_{z\to+\infty} \widehat{V}_1(z) = 0$, and $\lim_{z\to-\infty} \widehat{V}_2(z) = 0$. The columns of $\widehat{V}_{1,2}(z)$ matrices are the linearly independent solutions of the system (10) with a zero right hand side.

In order to find these linearly independent solutions we complete the Fourier transformation (7) in the Maxwell equations (3), as it was done for the Green's function. Excluding from equations components E_z, H_z and transiting to dimensionless variable $\xi = \alpha z$ we obtain a system of linear differential equations of the first order

$$\frac{\partial}{\partial \xi}\begin{pmatrix} E_x \\ E_y \\ H_x \\ -H_y \end{pmatrix} = -i\Omega \begin{pmatrix} 0 & 0 & 0 & 1 - \frac{q^2}{k_0^2 \varepsilon_\perp} \\ 0 & 0 & 1 & 0 \\ \varepsilon_{xy} & \varepsilon_{yy} - \frac{q^2}{k_0^2} & 0 & 0 \\ \varepsilon_{xx} & \varepsilon_{xy} & 0 & 0 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ H_x \\ -H_y \end{pmatrix}. \tag{14}$$

Here $\Omega = k_0/\alpha = 2d/\lambda$ is a large dimensionless parameter. In a matrix form the system (14) can be written as

$$\Phi'(\xi) = i\Omega\widehat{A}(\xi)\Phi(\xi)\,, \tag{15}$$

where $\widehat{A}(\xi + \alpha d) = \widehat{A}(\xi)$ is a periodic function. The solution of the system (15) with the initial condition $\Phi_0 = \Phi(\xi_0)$, $\xi_0 = \alpha z_0$ is

$$\Phi(\xi) = \widehat{M}(\xi, \xi_0)\Phi(\xi_0)\,, \tag{16}$$

where $\widehat{M}(\xi, \xi_0)$ is a matrix of evolution. In the first approximation over $1/\Omega$ the matrix of evolution $\widehat{M}(\xi, \xi_0)$ has the form [7]:

$$\widehat{M}(\xi, \xi_0) = \widehat{U}(\xi)\widehat{\Lambda}(\xi, \xi_0)\widehat{U}^{-1}(\xi_0)\,, \tag{17}$$

where $\widehat{\Lambda}(\xi, \xi_0)$ is a diagonal matrix with elements

$$\Lambda(\xi, \xi_0)_{ll} = \exp\left\{ \int_{\xi_0}^{\xi} \left[i\Omega\lambda_l(x) - \left(\widehat{U}^{-1}(x)\widehat{U}'(x)\right)_{ll} \right] dx \right\}. \tag{18}$$

The columns of the $\widehat{U}(\xi)$ matrix are eigenvectors of the $\widehat{A}(\xi)$ matrix, $\lambda_l(\xi)$ are corresponding eigenvalues. Eqs. (17), (18) are many-dimensional analog of WKB approximation. The structure of the solution (16), (17) agrees with the Floquet theorem [8, 9].

Using the special block structure of the $\widehat{A}(\xi)$ matrix, it is possible to get explicit expressions for $\widehat{\Lambda}$ and $\widehat{U}$ matrices [7]. As a result, for known components of electromagnetic field E_x^0, E_y^0, H_x^0, H_y^0 on the plane $z = z_0$ we get for components E_x, E_y, H_x, H_y in $(\mathbf{q}, z)$-representation

$$\begin{pmatrix} E_x \\ E_y \\ H_x \\ -H_y \end{pmatrix} = \frac{1}{2}\left[\begin{pmatrix} \hat{u}_{11}\exp(\hat{\phi}_+)\hat{u}_{110}^{-1} & \hat{u}_{11}\exp(\hat{\phi}_+)\hat{u}_{220}^{-1} \\ \hat{u}_{22}\exp(\hat{\phi}_+)\hat{u}_{110}^{-1} & \hat{u}_{22}\exp(\hat{\phi}_+)\hat{u}_{220}^{-1} \end{pmatrix}\right.$$
$$\left. + \begin{pmatrix} \hat{u}_{11}\exp(\hat{\phi}_-)\hat{u}_{110}^{-1} & -\hat{u}_{11}\exp(\hat{\phi}_-)\hat{u}_{220}^{-1} \\ -\hat{u}_{22}\exp(\hat{\phi}_-)\hat{u}_{110}^{-1} & \hat{u}_{22}\exp(\hat{\phi}_-)\hat{u}_{220}^{-1} \end{pmatrix}\right] \begin{pmatrix} E_x^0 \\ E_y^0 \\ H_x^0 \\ -H_y^0 \end{pmatrix}, \quad (19)$$

where

$$\exp\left(\hat{\phi}_\pm\right) = \sqrt{\frac{1 - \mathcal{H}\cos^2\zeta_0}{1 - \mathcal{H}\cos^2\zeta}}$$
$$\times \begin{pmatrix} \exp\left[\pm i\Omega\lambda_1(\xi - \xi_0)\right] & 0 \\ 0 & \sqrt{\frac{\lambda_2(\zeta_0)}{\lambda_2(\zeta)}}\exp\left[\pm i\Omega\int_{\xi_0}^{\xi}\lambda_2(x)\,dx\right] \end{pmatrix}, \quad (20)$$

and

$$\hat{u}_{11}(\xi) = \begin{pmatrix} \sin\zeta & (1-\mathcal{H})\cos\zeta \\ -\cos\zeta & \sin\zeta \end{pmatrix}, \quad \hat{u}_{110} = \hat{u}_{11}(\zeta_0),$$
$$\hat{u}_{22}(\xi) = \frac{1}{\mathcal{H}-1}\begin{pmatrix} 0 & 1-\mathcal{H} \\ 1 & 0 \end{pmatrix}\hat{u}_{11}(\xi)\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \hat{u}_{220} = \hat{u}_{22}(\zeta_0),$$
$$\lambda_1 = \sqrt{\varepsilon_\perp(1-\mathcal{H})}, \quad \lambda_2(\xi) = \sqrt{\varepsilon_\parallel - \varepsilon_{xx}\mathcal{H}},$$

Here $\zeta = \xi + \psi_0 \quad \zeta_0 = \xi_0 + \psi_0$.

Note, equations of the system (10) are derived from the system (14) if we exclude variables H_x and $-H_y$. Therefore it is possible to construct $\widehat{V}_1(z)$ and $\widehat{V}_2(z)$ matrices by selection the suitable solutions (E_x, E_y) in Eq. (19). We have

$$\widehat{V}_1(z) = \hat{u}_{11}(z)\exp\left(\hat{\phi}_+\right), \quad \widehat{V}_2(z) = \hat{u}_{11}(z)\exp\left(\hat{\phi}_-\right). \quad (21)$$

Then the expression (13) has the following form

$$\widehat{G}(z,z_1) = -\frac{\hat{u}_{11}(z)}{2ik_0}\sqrt{\frac{1-\mathcal{H}\cos^2\zeta_1}{1-\mathcal{H}\cos^2\zeta}}$$
$$\times\begin{pmatrix} \dfrac{\exp\left(ik_0\lambda_1\left|z-z_1\right|\right)}{\lambda_1} & 0 \\ 0 & \dfrac{\exp\left(ik_0\left|\int_{z_1}^{z}\lambda_2(x)\,dx\right|\right)}{\sqrt{\lambda_2(\zeta_1)\lambda_2(\zeta)}} \end{pmatrix}\hat{u}_{11}^{-1}(z_1), \quad (22)$$

where $\xi_1 = \alpha z_1$, $\zeta_1 = \alpha z_1 + \psi_0$. From Eqs. (12), (20) we get the Green's function $\widehat{T}(\mathbf{q}; z, z_1)$

$$\widehat{T}(\mathbf{q}; z, z_1) = \frac{\exp\left(ik_0\lambda_1\left|z-z_1\right|\right)\widehat{F}_1(\mathbf{q}; z, z_1)}{2ik_0\sqrt{(1-\mathcal{H}\cos^2\zeta_1)(1-\mathcal{H}\cos^2\zeta)}}$$
$$+\frac{\exp\left(ik_0\left|\int_{z_1}^{z}\lambda_2(x)\,dx\right|\right)\widehat{F}_2(\mathbf{q}; z, z_1)}{2ik_0\sqrt{(1-\mathcal{H}\cos^2\zeta_1)(1-\mathcal{H}\cos^2\zeta)}} + \widehat{F}_3(\mathbf{q}; z, z_1)\,, \quad (23)$$

where

$$\widehat{F}_1(\mathbf{q}; z, z_1) = \begin{pmatrix} \dfrac{(\mathcal{H}-1)\sin\zeta\sin\zeta_1}{\lambda_1} & \dfrac{(1-\mathcal{H})\sin\zeta\cos\zeta_1}{\lambda_1} & \dfrac{q\sin\zeta\sin\zeta_1}{k_0\varepsilon_\perp\,\mathrm{sign}(\xi-\xi_1)} \\ \dfrac{(1-\mathcal{H})\cos\zeta\sin\zeta_1}{\lambda_1} & \dfrac{(\mathcal{H}-1)\cos\zeta\cos\zeta_1}{\lambda_1} & \dfrac{q\cos\zeta\sin\zeta_1}{k_0\varepsilon_\perp\,\mathrm{sign}(\xi_1-\xi)} \\ \dfrac{q\sin\zeta\sin\zeta_1}{k_0\varepsilon_\perp\,\mathrm{sign}(\xi-\xi_1)} & \dfrac{q\sin\zeta\cos\zeta_1}{k_0\varepsilon_\perp\,\mathrm{sign}(\xi_1-\xi)} & \dfrac{\mathcal{H}\lambda_1\sin\zeta\sin\zeta_1}{\varepsilon_\perp(\mathcal{H}-1)} \end{pmatrix},$$

$$\widehat{F}_2(\mathbf{q}; z, z_1) = \lambda_2^{1/2}(\zeta)\lambda_2^{-1/2}(\zeta_1)$$
$$\times\begin{pmatrix} \dfrac{-(\mathcal{H}-1)^2\cos\zeta\cos\zeta_1}{\lambda_2(\zeta)} & \dfrac{(\mathcal{H}-1)\cos\zeta\sin\zeta_1}{\lambda_2(\zeta)} & \dfrac{(\mathcal{H}-1)q\lambda_2(\zeta_1)\cos\zeta\cos\zeta_1}{k_0\varepsilon_\perp\,\mathrm{sign}(\xi_1-\xi)\lambda_2(\zeta)} \\ \dfrac{(\mathcal{H}-1)\sin\zeta\cos\zeta_1}{\lambda_2(\zeta)} & \dfrac{-\sin\zeta\sin\zeta_1}{\lambda_2(\zeta)} & \dfrac{q\lambda_2(\zeta_1)\sin\zeta\cos\zeta_1}{k_0\varepsilon_\perp\,\mathrm{sign}(\xi-\xi_1)\lambda_2(\zeta)} \\ \dfrac{q(\mathcal{H}-1)\cos\zeta\cos\zeta_1}{k_0\varepsilon_\perp\,\mathrm{sign}(\xi_1-\xi)} & \dfrac{q\cos\zeta\sin\zeta_1}{k_0\varepsilon_\perp\,\mathrm{sign}(\xi-\xi_1)} & \dfrac{-\mathcal{H}\lambda_2(\zeta_1)\cos\zeta\cos\zeta_1}{\varepsilon_\perp} \end{pmatrix},$$

$$\widehat{F}_3(\mathbf{q}; z, z_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -k_0^{-2}\varepsilon_\perp^{-1}\delta(z-z_1) \end{pmatrix}$$

The first and the second terms in Eq. (23) correspond to the ordinary and extraordinary waves, the third term describes a longitudinal wave. The third term does not input a contribution to the asymptotics of the Green's function in the far zone.

The Green's function in the $\mathbf{r}$–representation has the form

$$\widehat{T}(\boldsymbol{\rho}-\boldsymbol{\rho}_1;z,z_1)=\int\frac{d\mathbf{q}}{(2\pi)^2}\exp\left[i\mathbf{q}(\boldsymbol{\rho}-\boldsymbol{\rho}_1)\right]\widehat{T}(\mathbf{q};z,z_1)\,. \qquad (24)$$

Let's analyse this function in the far zone, i.e., for $|\mathbf{r}-\mathbf{r}_1|\gg\lambda$. Then the integral (24) may be calculated using the stationary phase method. For this purpose it is necessary to find a stationary point $\mathbf{q}_{st}$ and to expand the exponent in Taylor series over $\mathbf{p}=(\mathbf{q}-\mathbf{q}_{st})$ up to terms of second order. In all nonexponential terms it is possible to put $\mathbf{q}=\mathbf{q}_{st}$. The remaining Gaussian integral is easily calculated

$$\int\exp\left(\frac{i}{2}\mathbf{p}\hat{H}\mathbf{p}\right)d\mathbf{p}=\frac{2\pi i\sigma}{\sqrt{|\det\hat{H}|}}\,. \qquad (25)$$

Here $\hat{H}$ is the real symmetric matrix of the second derivatives calculated at $\mathbf{q}=\mathbf{q}_{st}$. The factor σ in Eq. (25) depends on signs of real eigenvalues μ_1 and μ_2 of the $\hat{H}$ matrix: for $\mu_1>0$, $\mu_2>0$, $\sigma=1$; for $\mu_1\mu_2<0$, $\sigma=-i$; for $\mu_1<0$, $\mu_2<0$, $\sigma=-1$.

In the first term of Eq. (23) the stationary point is determined from the equation

$$\nabla_{\mathbf{q}}\left[k_0\lambda_1|z-z_1|+\mathbf{q}(\boldsymbol{\rho}-\boldsymbol{\rho}_1)\right]=0. \qquad (26)$$

Eq. (26) is easily solved and we get

$$\mathbf{q}_{st1}=\sqrt{\varepsilon_\perp}k_0\frac{\boldsymbol{\rho}-\boldsymbol{\rho}_1}{|\mathbf{r}-\mathbf{r}_1|}\,. \qquad (27)$$

The phase of the wave is equal to $\Psi_1=\sqrt{\varepsilon_\perp}k_0|\mathbf{r}-\mathbf{r}_1|$. It means, that a surface of wave vectors and a surface of a constant phase are spheres just as in the isotropic medium with permittivity $\varepsilon_\perp$.

The second term of Eq. (23) describes an extraordinary wave and the equation for the stationary point has the form

$$\boldsymbol{\rho}-\boldsymbol{\rho}_1=\frac{1}{k_0\varepsilon_\perp}\left|\int_z^{z_1}\frac{\hat{\varepsilon}^\perp(z')\mathbf{q}_{st2}\,dz'}{\lambda_2(z',\mathbf{q}_{st2})}\right|, \qquad (28)$$

where $\varepsilon^\perp_{\alpha\beta}(z)=\varepsilon_{\alpha\beta}(z)$, $(\alpha,\beta=1,2)$ is the transversal part of the $\hat{\varepsilon}(z)$ tensor, $\lambda_2(z,\mathbf{q})=\sqrt{\varepsilon_\parallel-(\mathbf{q}\hat{\varepsilon}^\perp(z)\mathbf{q})/k_0^2\varepsilon_\perp}$. The phase of the wave is equal to

$$\Psi_2=(\boldsymbol{\rho}-\boldsymbol{\rho}_1)\mathbf{q}_{st2}+k_0\left|\int_z^{z_1}dz'\lambda_2(z',\mathbf{q}_{st2})\right|. \qquad (29)$$

The Green's function in the far zone has the form

$$\widehat{T}(\mathbf{r},\mathbf{r}_1) = \frac{-\sqrt{\varepsilon_\perp}\exp(i\sqrt{\varepsilon_\perp}k_0\,|\mathbf{r}-\mathbf{r}_1|)\,|z-z_1|\,\widehat{F}_1(\mathbf{q}_{st1};z,z_1)}{4\pi\sqrt{(|\mathbf{r}-\mathbf{r}_1|^2-|\boldsymbol{\rho}-\boldsymbol{\rho}_1|^2\cos^2\zeta)(|\mathbf{r}-\mathbf{r}_1|^2-|\boldsymbol{\rho}-\boldsymbol{\rho}_1|^2\cos^2\zeta_1)}} - \frac{\varepsilon_\perp k_0\exp(i\Psi_2)(\det\widehat{A}(\mathbf{q}_{st2};z,z_1))^{-1/2}\widehat{F}_2(\mathbf{q}_{st2};z,z_1)}{4\pi\sqrt{(k_0^2\varepsilon_\perp-(\mathbf{q}_{st2}\mathbf{n}(z))^2)(k_0^2\varepsilon_\perp-(\mathbf{q}_{st2}\mathbf{n}(z_1))^2)}}, \quad (30)$$

where

$$A_{\alpha\beta}(\mathbf{q};z,z_1) = \frac{-1}{\sqrt{\varepsilon_\perp}}\left|\int_z^{z_1}\left[\frac{\varepsilon^\perp_{\alpha\beta}}{\sqrt{\varepsilon_\|\varepsilon_\perp k_0^2-(\mathbf{q}\hat{\varepsilon}^\perp\mathbf{q})}}+\frac{(\hat{\varepsilon}^\perp\mathbf{q})_\alpha(\hat{\varepsilon}^\perp\mathbf{q})_\beta}{\sqrt{\left(\varepsilon_\|\varepsilon_\perp k_0^2-(\mathbf{q}\hat{\varepsilon}^\perp\mathbf{q})\right)^3}}\right]dz'\right|.$$

4 WAVEGUIDE PROPAGATION

Let's find the stationary point $\mathbf{q}_{st2}$ for distances $|z-z_1|\gg d$. Note, the integrand in the right hand side of Eq.(28) is a periodic function. The primitive of a periodic function can be presented as a sum of a linear function and a periodic one [8]. The inclination of a linear part is equal to an average value on a period of the integrand. In the case of $|z-z_1|\gg d$ the periodic term can be omitted. We have

$$q_{st2} = \frac{2k_0^2\sqrt{\varepsilon_\|\varepsilon_\perp}\,|z-z_1|}{\pi\,|\boldsymbol{\rho}-\boldsymbol{\rho}_1|\,\sqrt{k_0^2\varepsilon_\perp-q_{st2}^2}}\left|K\left(\sqrt{\frac{\varepsilon_a q_{st2}^2}{\varepsilon_\|(q_{st2}^2-k_0^2\varepsilon_\perp)}}\right) + \frac{1}{k_0^2\varepsilon_\perp}(q_{st2}^2-k_0^2\varepsilon_\perp)E\left(\sqrt{\frac{\varepsilon_a q_{st2}^2}{\varepsilon_\|(q_{st2}^2-k_0^2\varepsilon_\perp)}}\right)\right|, \quad (31)$$

where

$$K(a) = \int_0^{\frac{\pi}{2}}\frac{d\psi}{\sqrt{1-a^2\sin^2\psi}}, \quad E(a) = \int_0^{\frac{\pi}{2}}\sqrt{1-a^2\sin^2\psi}\,d\psi \quad (32)$$

are the Legendre normal elliptic integrals [10]. The phase of the wave is equal to

$$\Psi_2 = |\boldsymbol{\rho}-\boldsymbol{\rho}_1|q_{st2} + (z-z_1)q_{z2}\,, \quad (33)$$

where

$$q_{z2} = \frac{2\,\mathrm{sign}(z - z_1)}{\pi} \sqrt{k_0^2\varepsilon_{\|} - \frac{\varepsilon_{\|}}{\varepsilon_{\perp}} q_{st2}^2}\, E\left(\sqrt{\frac{\varepsilon_a q_{st2}^2}{\varepsilon_{\|}(q_{st2}^2 - k_0^2\varepsilon_{\perp})}}\right). \quad (34)$$

Note, functions $K(a)$ and $E(a)$ are real for $-1 < a < 1$ and for pure imaginary a. Hence the wave vector $\mathbf{k}_{st2} = (\mathbf{q}_{st2}, q_{z2})$ is real if the following condition is valid

$$q_{st2}^2 < k_0^2 \min\left(\varepsilon_{\perp}, \varepsilon_{\|}\right). \quad (35)$$

For other values of q_{st2} the phase Ψ_2 becomes complex and the wave does not propagate in the region of large $|z - z_1|$ due to damping. For these values of q_{st2} a forbidden zone is formed, i.e., there is a restriction on possible directions of the wave vector $\mathbf{k}_{st2}$.

Figure 1 shows the cross section of a surface of wave vectors by a plane containing the z axis for $\varepsilon_a > 0$. The discontinuities correspond to the forbidden zone. Note, the directions of the wave and the beam vectors in such a medium do not coincide, as well as in a uniaxial anisotropic medium [11].

If the condition (35) is violated, a ray begins to turn and the component q_{z2} becomes equal to zero in some point $z = z_t$, then it changes the sign. It means that a ray as though reflects by a plane $z = z_t$. As a refractive index is a periodic function, such a ray will be reflected alternately by two planes. Thus, a planar wave channel is formed [12, 13]. Inside this channel the waves can propagate at large distances ρ remaining within the limits of one period of z.

In order to explain the origin of the wave channel, we consider an extraordinary wave with the wave vector $\mathbf{k}(z) = k_0 n\mathbf{t}$, lying in the plane y–z: $\mathbf{k}(z) = (0, k_{\perp}, k_z(z))$, $\mathbf{t} = (0, \sin\chi(z), \cos\chi(z))$, $\chi(z)$ is the angle between the wave vector and the z axis, n is the refractive index. Note, due to the Snells law $k_{\perp} = k_0 n \sin\chi(z)$ remains constant. The refractive index of an extraordinary wave has the form

$$n = \sqrt{\frac{\varepsilon_{\|}\varepsilon_{\perp}}{\varepsilon_{\|}\cos^2\theta + \varepsilon_{\perp}\sin^2\theta}}, \quad (36)$$

where $\cos\theta = (\mathbf{n}, \mathbf{t}) = \sin(\alpha z + \psi_0)\sin\chi(z)$ is the angle between the wave vector and the optical axis.

Let the wave be radiated from the origin of coordinates at the angle χ_0 to the z axis. By virtue of the Snells law

$$k^2(z)\sin^2\chi(z) = k^2(0)\sin^2\chi_0 \quad (37)$$

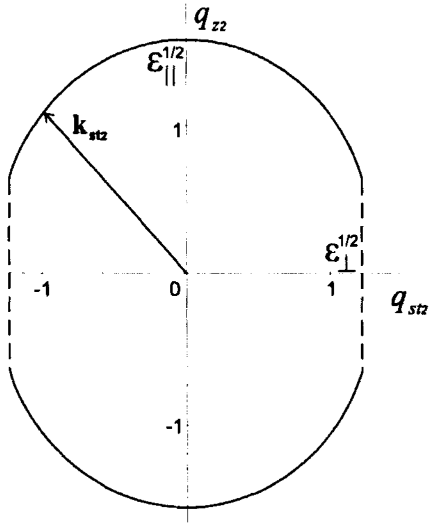

FIGURE 1

A cross section of a surface of wave vectors by a plane, containing z axis, for $\varepsilon_a = 1.0$, $\varepsilon_{\|} = 2.5$. The dotted line shows the forbidden zones. The value of k_{st2} depends on the wave vector direction. All wave numbers are expressed in terms of k_0.

we have

$$\sin^2\chi(z) = \frac{\varepsilon_{\perp}\sin^2\chi_0}{\varepsilon_{\perp} + \varepsilon_a \sin^2\chi_0(\sin^2\psi_0 - \sin^2(\alpha z + \psi_0))}. \tag{38}$$

In the turn point $k_z(z_t) = 0$, the angle $\chi(z_t) = \pi/2$. Then we can obtain from Eq. (38)

$$\sin^2(\alpha z_t + \psi_0) = \varepsilon_{\perp}\varepsilon_a{}^{-1}\cot^2\chi_0 + \sin^2\psi_0\,. \tag{39}$$

Taking into account in Eq. (39) the condition $0 \leqslant \sin^2(\alpha z_t + \psi_0) \leqslant 1$, we come to restriction (35), $k_{\perp}^2 < k_0^2 \min\left(\varepsilon_{\perp}, \varepsilon_{\|}\right)$.

Let's consider the trajectory of a beam $\mathbf{r}(z) = (x(z), y(z), z)$. The beam vector $\mathbf{S}$ of an extraordinary wave lies in the same plane as vectors $\mathbf{k}$ and $\mathbf{n}$:

$$\mathbf{S} = S\mathbf{s}\,, \quad \mathbf{s} = \frac{\varepsilon_{\perp}\mathbf{t} + \varepsilon_a\cos\theta\,\mathbf{n}}{\sqrt{\varepsilon_{\perp}^2\sin^2\theta + \varepsilon_{\|}^2\cos^2\theta}}. \tag{40}$$

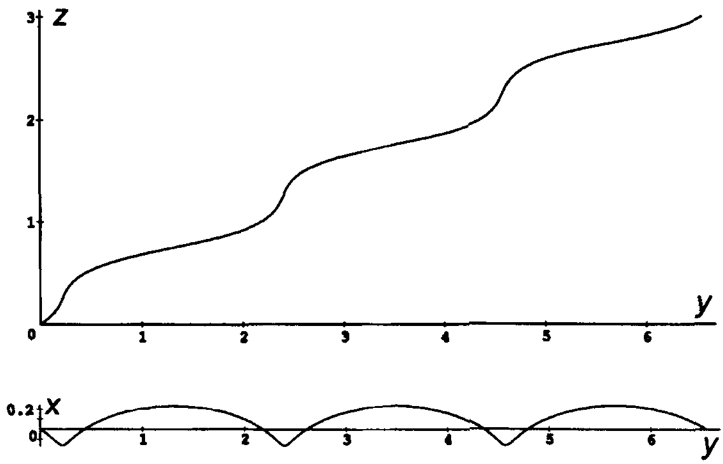

FIGURE 2

The projections of a beam in z–y and x–y planes outside the wave channel. Trajectory was calculated for $\varepsilon_a = 2.0$, $\varepsilon_{\|} = 2.5$, $\psi_0 = -\pi/4$, $\chi_0 = \pi/6$. All distances are expressed in terms of d.

Vector **S** is parallel to a tangent to the beam trajectory in every point. It leads to relations

$$x'(z) = s_x(z)\ell(z),\ y'(z) = s_y(z)\ell(z),\ 1 = s_z(z)\ell(z)\,, \tag{41}$$

where $\ell(z) = \sqrt{x'^2(z) + y'^2(z) + 1}$. Then we get from Eqs. (40), (41)

$$\begin{aligned} x'(z) &= \frac{\varepsilon_a}{\varepsilon_\perp}\sin(\alpha z + \psi_0)\cos(\alpha z + \psi_0)\tan\chi(z)\,, \\ y'(z) &= \tan\chi(z) + \frac{\varepsilon_a}{\varepsilon_\perp}\sin^2(\alpha z + \psi_0)\tan\chi(z)\,, \end{aligned} \tag{42}$$

where $\tan\chi(z)$ is obtained from Eq. (38)

$$\tan\chi(z) = \left[\cot^2\chi_0 + \varepsilon_a{\varepsilon_\perp}^{-1}\left(\sin^2\psi_0 - \sin^2(\alpha z + \psi_0)\right)\right]^{-1/2}. \tag{43}$$

The trajectory of the beam can be found by integrating Eqs.(42). Outside the forbidden zone the trajectory is calculated at once by integration Eqs.(42) (See Figure 2). Inside the wave channel it is

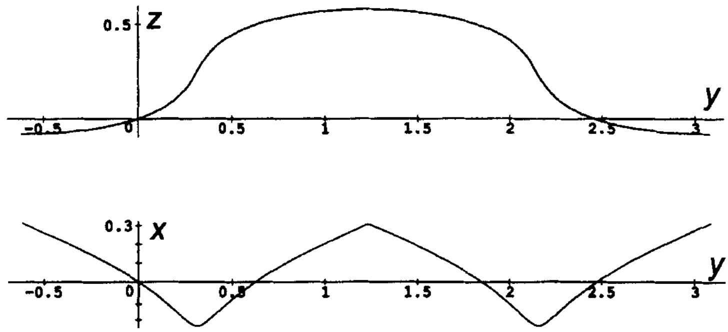

FIGURE 3

The projections of a beam in z–y and x–y planes inside the wave channel. Trajectory was calculated for $\varepsilon_a = 2.0$, $\varepsilon_{\|} = 2.5$, $\psi_0 = -\pi/4$, $\chi_0 = \pi/4$. All distances are expressed in terms of d.

necessary to sew correctly regions with positive and negative k_z (See Figure 3). One can see, that the trajectory of an extraordinary beam is not plane inside and outside of the wave channel.

The asymptotics of the Green's function inside and outside of the wave channel are different. Outside of the wave channel the Green's function behaves as $|\mathbf{r}-\mathbf{r}_1|^{-1}$ for $|\mathbf{r}-\mathbf{r}_1| \to \infty$. Inside the wave channel waves propagate within one layer at any $|\boldsymbol{\rho}-\boldsymbol{\rho}_1|$. Therefore the energy density of these waves decreases as $|\mathbf{r}-\mathbf{r}_1|^{-1} \approx |\boldsymbol{\rho}-\boldsymbol{\rho}_1|^{-1}$. Hence the amplitude of the field of a point source behaves as $|\mathbf{r}-\mathbf{r}_1|^{-1/2}$ at $|\mathbf{r}-\mathbf{r}_1| \to \infty$ when points z and z_1 are in the same wave channel.

Acknowledgments

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